Stochastic resonance for quantum channels

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The concept of stochastic resonance in nonlinear dynamics is applied to interpret the capacity of noisy quantum channels. The two-Pauli channel is used to illustrate the idea. The fidelity of the channel is also considered. Noise enhancement is found for the channel fidelity but not for the channel capacity. [S1063-651X(99)14003-0]

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I. INTRODUCTION

Stochastic resonance is a phenomenon concerned with amplifying a small signal forcing a nonlinear system by addition of a stochastic noise to the signal [1]. The physics behind this phenomenon is the transfer of energy from the stochastic field into some physical process with the assistance of the signal.

Most earlier works were concerned with periodical signals. On the other hand, it was pointed out by Moss in 1989 that one may associate the switching events in a stochastic bistable and threshold system with an information flow through the system [2]. To consider an aperiodical signal for a channel performance, the peak of mutual information between the input signal and output signal is used as the definition of resonance [3–5], because of the informax ansatz, which uses the mutual information to assess different ways of information processing, and because, traditionally, the resonance condition is defined as the peak of the output signal to noise ratio for the periodical signal case.

However, previous considerations for aperiodical signals are for classical channels. Stochastic resonance has been studied for quantum systems [1]. It is known, for periodical signal cases, that quantum mechanics can provide additional routes by quantum tunneling to overcome a threshold. The classical stochastic resonance effect can be enhanced up to two orders of magnitude for strongly damped systems. Therefore it will be interesting to see whether, such resonance also exits in quantum channels.

Recently, because of the development of quantum computers [6] people have become interested in information transmission through noisy quantum channels [7]. It can be used to describe processes such as computer memory or other secondary storage, quantum cryptography [8], and quantum teleportation [9]. To study the noise enhanced channel capacities of such channels one needs a measure for noise and a measure for the channel capacities, or a measure for any other property interested. Therefore, the first problem need to answer is: Which quantity can be used as the *correct* measure for the resonance?

There are several such measures, analogous to the classical Shannon's mutual information, emerging during the study of quantum computing [10–12]. In this paper Schumacher's formulation of coherent information is followed.

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II. NOISY CHANNEL MODEL

A quantum channel can be considered as a process defined by an input density matrix ρ_x, and an output density matrix ρ_y, with the process described by a quantum operation $\mathcal{N}$,

$$\rho_x \overset{\mathcal{N}}{\longrightarrow} \rho_y . \tag{1}$$

Because of decoherence, the superoperator $\mathcal{N}$ is not unitary. However, on a larger quantum system that includes the environment E of the system, the total evolution operator U_{xE} become unitary. The environment may be considered to be initially in a pure state $|0_E\rangle$ without loss of generality. In this case, the superoperator can be written as

$$\mathcal{N}(\rho_x) = \mathrm{Tr}_E U_{xE}(\rho_x \otimes |0_E\rangle\langle 0_E|) U_{xE}^\dagger . \tag{2}$$

The partial trace Tr_E is taken over environmental degree of freedom. Equation (2) can be rewritten as

$$\mathcal{N}(\rho_x) = \sum_i A_i \rho_x A_i^\dagger , \tag{3}$$

in which the A_i satisfy the completeness relation

$$\sum_i A_i^\dagger A_i = I, \tag{4}$$

which is equivalent to requiring $\mathrm{Tr}[\mathcal{N}(\rho_x)] = 1$. Conversely, any set of operators A_i satisfying Eq. (4) can be used in Eq. (3) to give rise to a valid noisy channel in the sense of Eq. (2). The mutual information in the classical formalism becomes

$$H(x:y) = H(\rho_x) + H(\mathcal{N}(\rho_x)) - H_e(\rho_x, \mathcal{N}), \tag{5}$$

in which $H(\cdot) = -\mathrm{Tr}\cdot\log_2\cdot$ is the von Neumann entropy [13], and

$$H_e(\rho_x, \mathcal{N}) \equiv -\mathrm{Tr}(W \log_2 W) \tag{6}$$

with

$$W_{ij} \equiv \mathrm{Tr}(A_i \rho_x A_j^\dagger) \tag{7}$$

measures the amount of information exchanged between the system x and the environment E during their interaction [7], which can be used to characterize the amount of quantum

noise N in the channel. If the environment is initially in a pure state, the entropy exchange is just the environment's entropy after the interaction.

The coherent information is defined as

$$C(\rho_x,\mathcal{N}) \equiv H\left(\frac{\mathcal{N}(\rho_x)}{\mathrm{Tr}(\mathcal{N}(\rho_x))}\right) - N(\rho_x,\mathcal{N}), \tag{8}$$

which plays a role in quantum information theory analogous to that played by the mutual information in classical information theory. It can be positive, negative, or zero. Furthermore, it is a function of the input state and the channel only. Although the coherence information is generally believed to represent only a lower bound on the channel capacity in Shannon's definition, it can be used to represent the channel capacity without talking about encoding [14].

In what follows this C-N relationship is demonstrated by the two-Pauli channel [15].

III. TWO-PAULI CHANNEL

The *two-Pauli channel* is a noisy quantum channel on a single qubit with

$$A_1 = \sqrt{x}I, \quad A_2 = \sqrt{\tfrac{1}{2}(1-x)}\sigma_1, \quad A_3 = -i\sqrt{\tfrac{1}{2}(1-x)}\sigma_2, \tag{9}$$

where I is the identity matrix and σ_1,σ_2, and σ_3 are the Pauli matrices, i.e.,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10}$$

This channel has a simple interpretation: with probability x, it leaves the qubit alone; with probability $1-x$ it randomly applies one of the two Pauli rotations to the qubit.

A general (input) state in the Bloch sphere representation can be written as

$$\rho_x = \tfrac{1}{2}(I + \vec{a}\cdot\vec{\sigma}). \tag{11}$$

Here, $\vec{a} = (a_1,a_2,a_3)$ is the Bloch vector of length unity or less, and $\vec{\sigma}$ is the vector of Pauli matrices. For two-state systems, a Bloch vectors with unity radius describe pure quantum states, those with radius less than unity described mixed states, and those with radius greater than unity do not describe any quantum state. The action of the channel on this density matrix is

$$\mathcal{N}(\rho_x) = \tfrac{1}{2}(I + \vec{b}\cdot\vec{\sigma}), \tag{12}$$

in which

$$\vec{b} = (a_1 x, a_2 x, a_3(2x-1)). \tag{13}$$

The matrix W thus computed reads

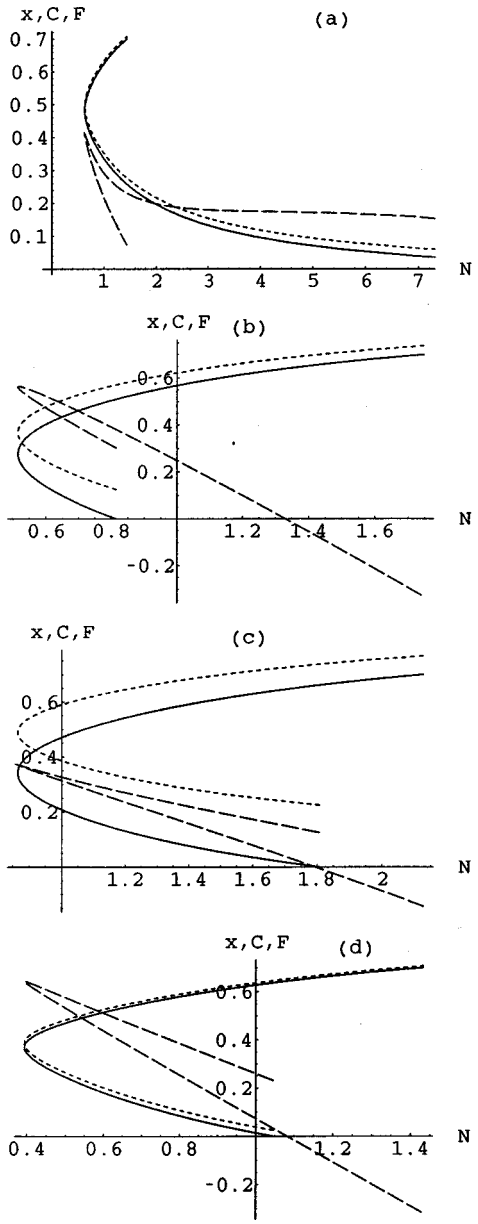

FIG. 1. Parametric plots of retention rate x vs noise N (solid line), coherence information C vs noise N (long dashed lines), and fidelity F vs noise N (short dashed lines) for the parameter x from 0.0 to 0.7 and various initial states: (a) $a_1=0.1, a_2=0.2, a_3=0.9$; (b) $a_1=0.3, a_2=0.4, a_3=0.2$; (c) $a_1=0.6, a_2=0.3, a_3=0.5$; (d) $a_1=0.1, a_2=0.2, a_3=0.3$.

$$W = \begin{pmatrix} x & a_1\sqrt{\dfrac{x(1-x)}{2}} & ia_2\sqrt{\dfrac{x(1-x)}{2}} \\ a_1\sqrt{\dfrac{x(1-x)}{2}} & \dfrac{1-x}{2} & \dfrac{a_3(1-x)}{2} \\ -ia_2\sqrt{\dfrac{x(1-x)}{2}} & \dfrac{a_3(1-x)}{2} & \dfrac{1-x}{2} \end{pmatrix}. \tag{14}$$

The noise strength

$$N = -\sum_{i=1}^{3} \lambda_i \log_2 \lambda_i \tag{15}$$

with λ_i been the eigenvalues of the W matrix, while

$$H(\rho_y) = -\sum_{i=1}^{2} \theta_i \log_2 \theta_i , \tag{16}$$

with $\theta_{1,2} = [1 \pm \sqrt{(a_1^2 + a_2^2)x^2 + a_3^2(1-2x)^2}]/2$. The noise enhancement can be investigated by looking for some initial states (a_1, a_2, a_3) and flipping rate x, where the slope of $\partial C/\partial N > 0$. However, in practice it is difficult to calculate such function analytically and obtain useful results. Some examples of the capacity-noise relation are plotted in Fig.1. For all cases we tried we find no such enhancement. However, at some range of noise the capacity is not a single valued function. With a proper choice of the flipping rate one can indeed have higher capacity. This is a result of the flipping rate x been a nonmonotonic function of the noise as shown in the solid lines of Fig. 1. A moderate flipping rate actually reduce the noise for all cases plotted. For a communication channel the (entangled) fidelity,

$$F = \sum_{\mu} (\mathrm{Tr}\, \rho_x A_\mu)(\mathrm{Tr}\, \rho_x A_\mu^\dagger), \tag{17}$$

is also of our concern, since it represent the quality of the signal transmitted. For the two-Pauli channel

$$F = \tfrac{1}{2}(a_1^2 + a_2^2)(1-x) + x, \tag{18}$$

they are plotted along with the coherent information in Fig. 1. The fidelity do exhibit noise enhancement clearly as shown in Figs. 1(b)–1(d). For the best performance of a channel, there are some tradeoffs between the fidelity and the capacity. It is interesting to notice the capacity-noise curve becomed cusp and eventually collapses into a line as one approaches the Bloch sphere. Note that, some people might think fidelity can be used as a measure of the noise strength. However, it is only an *indirect* measure. It measures the effect of the noise instead of the noise itself. As shown in the figures, the fidelity is close to the noise in some cases.

IV. CONCLUSION

In summary, ''quantum stochastic resonance'' in a quantum communication channel is considered in the present work. A ''resonance'' of the channel capacity is not found in the two-Pauli channel. On the other hand, they do have noise enhanced fidelity. However, the classical stochastic resonance phenomenon is a result of interplay between probabilistic and deterministic evolutions. Such interplay is manifested in the coherence of quantum states. As the dissipation enhanced quantum channel capacity is consistent with the results found in periodical forced quantum systems, it is very possible for stochastic resonance of both fidelity and capacity to exits on other types of sources, channels and even definition of capacity and fidelity. There might have subtlety among different channels, as people have found in the cases of stochastic resonances that various potential wells will result in slightly different resonance conditions. They are under further investigation.

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